

# On Dust Charging Equation

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A general derivation of the charging equation of a dust grain is presented, and indicated where and when it can be used. A problem of linear fluctuations of charges on the surface of the dust grain is discussed.

In recent years a huge number of works have been devoted to the investigation of dusty plasma and a dust in plasmas, taking into account the charging equations. However, to the best of our knowledge, in the literature the derivation of the charging equation is missing. Also there does not exist any work where the definition of surface charge and total current of electrons and ions is accurately given.

Another flaw arises in the literature in considerations of fluctuations of the surface charge and the total currents. Namely, an inappropriate equations have been used in the studies of perturbations of the surface charge and the total currents yielding the damping of perturbations in a collisionless plasmas, which is physically incorrect. Thus, the general derivation of the charging equation of dust grain and the accurate definition of the surface charge and the total currents remain unsolved problems of the high importance.

In this Letter, we derive the charging equation from the Maxwell's equations, without any assumptions, and indicate the validity of this equation, that is when and where it can be used. We discuss about the surface charge and the total currents of electrons and ions in order to set right the use of equations for them. We show that a small perturbations of the number of charge on surface in the collisionless plasma, if the Landau damping effect is ignored, once appear they are undamped, because there is no mechanism of it.

We remind here readers that in Electrodynamics charges are treated as points and the charge of a particle is an invariant quantity, that is it does not depend on the choice of reference system. This means that, when one derives the continuity equation and the equation of motion of charged particles by the Lagrange equations, one should assume that the charge is constant. Otherwise the Lorentz force will have quite another form, so that even for one particle case the classical field theory will breakdown. To elucidate this, we use the Lagrangian for the charge of dust grain in an electromagnetic (EM) field [1]

$$L = -m_D c^2 \sqrt{1 - v^2/c^2} + \frac{Z_D}{c} (\vec{A} \cdot \vec{v}) - Z_D \varphi, \quad (1)$$

where  $m_D$  is the mass of the dust grain,  $\vec{A}$  is the three dimensional vector potential of EM field and  $\varphi$  is the scalar potential.

With the Lagrangian (1) at hand, as is well known one can simply obtain the equations of motion of a charge in EM field by employing the Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \vec{v}} = \frac{\partial L}{\partial \vec{r}}. \quad (2)$$

The result is

$$\frac{d\vec{p}}{dt} = Z_D \left( \vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right) + \vec{F}, \quad (3)$$

where the charge  $Z_D$  is supposed to vary in time and space, the expression in bracket on the right-hand side is the usual Lorentz force,  $\vec{F}$  is the additional force due to the variation of charge of the dust grain and has such form

$$\vec{F} = \left( \vec{A} \cdot \vec{v} - \varphi \right) \nabla Z_D - \frac{\vec{A}}{c} \left( \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) Z_D. \quad (4)$$

It is obvious that the additional force  $\vec{F}$  is nonphysical and it is not defined, because of violation of the Gauge invariance. Therefore, one must assume that the total charge on surface of the dust grain is constant in time and space.

Moreover Eqs. (1),(3) and (4) indicate that the Lagrangian used above does not exactly describe the interaction of the changeable charge with EM field. Hence, for the latter a new Lagrangian should be introduced, from which can be derived equations of motion that meet the Gauge invariance.

The same is, when we want to get the Maxwell's equations defined by a total action for particles plus field in the Gaussian system of units, which has the form [1]

$$S = - \sum_{\alpha} \int m_{\alpha} c dS_{\alpha} - \frac{1}{c^2} \int A_i j^i d\Omega - \frac{1}{16\pi c} \int F_{ik} F^{ik} d\Omega , \quad (5)$$

where  $\alpha$  denotes the particle species,  $j^i$  is the total current density and the other notation is standard.

We specifically note here that in Eq.(5) it is assumed that the charge on any species is constant. Therefore, employing any equation of the macroscopic electrodynamics one must suppose that charges of dust grains are constant, i.e., we may conclude that despite extensive theoretical efforts, there is still no electrodynamics for the macroscopic bodies with the changing charge.

The question, which we discuss in the following, concerns only one dust grain in a plasma. Let us start with the definition of surface charge on the dust grain. As is well known [2],[3] the distribution of excess charge of a conducting body lies entirely on the surface of a conductor. Since  $E = 0$  on the inner area of the surface, that is no electric field inside the grain, the tangential component of the electric field at the surface is zero. But there is a normal component  $E_n = \vec{E} \cdot \vec{n}$  (where  $\vec{n}$  is the unit vector) of the electric field just outside the surface. The normal component of the electric field takes very large values in the immediate neighborhood of the surface. It is very important to emphasize that the normal component  $E_n$  pertains to the surface itself, however it is not involved in the volume electrostatic problem, because it falls off over distances comparable with the distances between atoms.

Expression of the normal component  $E_n$  can readily be obtained by integration of the Poisson's equation. The result is

$$E_n = 4\pi \int \rho dn = 4\pi\sigma , \quad (6)$$

where  $\sigma$  and  $\rho$  are the surface and the volume charge densities, respectively.

Equation (6) can be written in such form

$$\sigma = -\frac{1}{4\pi} \frac{\partial \varphi}{\partial n} , \quad (7)$$

where the electrostatic potential  $\varphi$  is constant on the surface and fast decreases along the normal to the surface.

Using Eq.(7), we define the expression for the total charge on the surface of one dust grain as

$$Z_D = \int \sigma dS = -\frac{1}{4\pi} \int \frac{\partial \varphi}{\partial n} dS , \quad (8)$$

where the integral is taken over the whole surface, and  $dS$  is the element of area on the surface. Note that the equation (8) reads just the Gauss theory. Namely, the Poisson's equation in an integral form is

$$\oint \vec{E} d\vec{S} = \oint (\vec{E} \cdot \vec{n}) dS = 4\pi \int \sigma dS = 4\pi Z_D . \quad (9)$$

We now derive the charging equation from the Maxwell's equation

$$\text{curl} \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \vec{j} , \quad (10)$$

where  $\vec{j}$  is the total current density and can be expressed as

$$\vec{j} = \sum_{\alpha} e_{\alpha} n_{\alpha} \vec{v}_{\alpha} = \sum_{\alpha} e_{\alpha} \int d^3v \vec{v} f_{\alpha} . \quad (11)$$

Integrating Eq.(10) over surface of the grain and applying Stoke's theorem

$$\int \text{curl} \vec{B} d\vec{S} = \oint \vec{B} d\vec{l}$$

we get

$$\oint \vec{B} d\vec{l} = \frac{1}{c} \frac{\partial}{\partial t} \int \vec{E} d\vec{S} + \frac{4\pi}{c} \int \vec{j} d\vec{S} . \quad (12)$$

Noting that the circulation of magnetic field around any contour for the closed surface equals zero, and using the definition (8),(9) of the total charge on the surface of one grain, we finally obtain the charging equation in the form

$$\frac{\partial Z_D}{\partial t} = - \sum_{\alpha} \int \vec{j}_{\alpha} d\vec{S} = - \sum_{\alpha} \int (\vec{j}_{\alpha} \cdot \vec{n}) dS = - \sum_{\alpha} I_{\alpha} , \quad (13)$$

where  $I_{\alpha}$  is the total current of electrons and ions. Vector  $d\vec{S}$  is directed, as always, along the outward normal to the surface of the grain, that is along the normal towards the outside of the volume under consideration. It should be emphasized that a left-hand side of Eq.(13) is the partial derivative in time, but not the total derivative with respect to time.

Note that the equation (13) is the exact expression of the total charge on the surface of one grain for the case, when the surface tension [4],[5] is not taken into account. Detailed analysis of a charged surface is given in Ref.[6].

We now analyze the charging equation and explain how one should understand the total current. In Eq. (13) the current density has the normal component  $j_n$  alone, and differs from zero even at Maxwell-Boltzmann distribution function. However, well away from the grain  $j_n = 0$ , the equation (13) illustrates just one thing that the total number of charges on the surface can change only if the total current of electrons  $I_e$  is less or more than the total current of ions  $I_i$ . Such situation we have during charging processes of surface and this means that the equation (13) can be used for the charged surface alone.

Let us in more detail consider the total current of particles, which is

$$I_{\alpha} = \sum_{\alpha} \int d\vec{S} \vec{j}_{\alpha} = \int dS (\vec{n} \cdot \vec{j}) = e_{\alpha} \int dS \int d^3v v_n f_{\alpha} . \quad (14)$$

The distribution function of particles  $f_{\alpha}$  in general is function of all space coordinates, time and momentum, i.e.,  $f_{\alpha}(\vec{p}, \vec{r}, t) \rightarrow f(\vec{p}, \varphi(\vec{r}, t), \vec{A}(\vec{r}, t))$ . It follows that in order to define the total current, one must integrate right-hand side of Eq.(14) over the whole surface. Above explanation about the expression (14) allows us to conclude that the orbit-limited motion (OLM) approximation [7] is valid on the surface of dust grain, where the total current of electrons and ions balance each other.

We next discuss the question of linearization of the charging equation, small perturbations of the surface charges and damping of these perturbations. We specifically note here that too many publications were devoted to the study of damping of various plasma waves in a collisionless plasmas (ignoring the Landau damping) due to the fluctuation of the total surface charge. But nobody discussed the mechanism driving to the damping. As we already mentioned the charging equation describes any processes on the surface of one grain. Equation (13) exhibits that the total surface charge changes in time (increases or decreases on surface) depending on  $I_e < I_i$  or  $I_e > I_i$ . So that we have strictly nonstationary processes. Note that if we evoke a small perturbation  $Z_D = Z_{D0} + \delta Z_D$ , then the average  $\langle \delta Z_D \rangle = 0$  and the total charge  $\langle Z_D \rangle$  remains constant. If  $\delta Z_D$  is considered in linear approximation, then once appear, it will remain on the surface without damping. In this case one can talk about the surface waves on the surface of one grain [6]. As is well known the surface waves in the collisionless medium are undamped.

As an example, we will show by a simple model (without losing generality) that the fluctuation of the total surface charge cannot lead to damping of the surface waves. To this end, we write the variation of the total current  $I = I_e + I_i$  in such form

$$\delta I = \sum_{\alpha} \delta I_{\alpha} = \sum_{\alpha} \delta \int dS (\vec{n} \cdot \vec{j}_{\alpha}) \approx \sum_{\alpha} 4\pi r_g^2 \delta j_{n\alpha} = 4\pi r_g^2 \sum_{\alpha} e_{\alpha} n_{0\alpha} v_{n\alpha} , \quad (15)$$

where  $r_g$  is the radius of dust grain,  $n_{0\alpha}$  is the equilibrium density. Since  $v_{n\alpha}$  is the normal component of velocity, then  $v_{n\alpha} = \frac{\partial \delta r_{n\alpha}}{\partial t}$ . So, the equation (13) for the perturbation reads  $\frac{\partial \delta Z_D}{\partial t} = - \sum_{\alpha} \beta_{\alpha} \frac{\partial \delta r_{n\alpha}}{\partial t}$  or

$$\delta Z_D = - \sum_{\alpha} \beta_{\alpha} \delta r_{n\alpha} , \quad (16)$$

where  $\beta_{\alpha} = 4\pi r_g^2 e_{\alpha} n_{0\alpha}$  is a constant, and  $\delta r_{n\alpha}$  is proportional to the variation of the surface charge  $\sigma$ .

The equation of motion for the normal component of velocity on the surface can be written as

$$\frac{\partial^2 \delta r_{n\alpha}}{\partial t^2} = \frac{e_{\alpha} E_n}{m_{\alpha}} . \quad (17)$$

Use of Eq.(6) in Eq.(17) for the linear perturbations  $\delta \sigma \sim e^{-i\omega t}$  yields the solution

$$\delta r_{n\alpha} = - \frac{4\pi e_{\alpha}}{m_{\alpha} \omega^2} \delta \sigma . \quad (18)$$

Substituting Eq.(18) into Eq.(16) we finally obtain the following expression for the variation of the total charge on the surface

$$\delta Z_D = \frac{\sum_{\alpha} \omega_{p\alpha}^2}{\omega^2} 4\pi r_g^2 \delta\sigma ,$$

where  $\omega_{p\alpha}$  is the plasma frequency of the particle species  $\alpha$ . Note that the same result we can get from the kinetic equation if we neglect the Landau damping effect.

In summary, we have derived the charging equation of a dust grain immersed in a plasma and have shown when and where one can use it. We have discussed the electrodynamic properties of the dusty plasma, and concluded that a charge on the dust grain must be considered as a point particle with the constant charge, because today does not exist the electrodynamics of medium with variable charge of particles.

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